Topology and its Applications 35 (1990) 197-207
North-Holland

ON BOREL GROUPS

Fons van ENGELEN

Faculteit Der Economische Wetenschappen, Erasmus Universiteit Rotterdam, 3000 DR Rotterdam, Netherlands

Received 18 June 1988
Revised 16 December 1988

We investigate the question of which homogeneous (zero-dimensional absolute) Borel sets can have the structure of a topological group. We show that a Baire such space can only be a group if it is topologically complete. Then, using independent subsets, it is proved that if a homogeneous first category Borel set is homeomorphic to its own square, then it can be embedded as a topological subgroup of the Cantor group. Finally, it is shown that many first category homogeneous Borel sets cannot be given the structure of a topological group.

AMS (MOS) Subj. Class.: Primary 54H05; secondary 03E15, 22A05, 54H10

homogeneous absolute Borel set topological group

Introduction

All spaces considered are separable and metrizable.

Being a contribution to the "Eric van Douwen memorial issue" of Topology and its Applications, this paper deals with questions that Eric has been very interested in, having worked on related problems himself. As noted in the *In Memoriam*, written by van Mill [12], Eric was the first topologist to construct an example of a homogeneous zero-dimensional (separable metrizable) space T which is not a group. Subsequently, van Mill [11] gave another such example, his space S. The homogeneity of S and T is derived from their topological characterizations, obtained by rather involved arguments (see van Mill [11], van Engelen and van Mill [3]). Having characterized all homogeneous zero-dimensional absolute Borel sets in [2], I tried to find other examples of nongroups among them, as well as examples of groups. The results I obtained are contained in this paper. As it turns out, the "nongroupity" of S and T is a special case of the more general result, proved in Section 2 of this paper, that an absolute Borel group that is Baire must be complete. In Section 3 we show that there are also plenty of homogeneous zero-dimensional absolute Borel sets X that do admit the structure of a group: in fact, if such X is homeomorphic to X^2, and X is first category, then X is a subgroup of 2^ω. But there also exist zero-dimensional homogeneous absolute Borel sets X that are first category and not homeomorphic to X^2, e.g. $Q\times T$ is not homeomorphic to $Q\times T^2$. In Section 4 we show that $Q\times T$ is not a group; in fact, the same method will give countably many first category homogeneous zero-dimensional absolute Borel nongroups.

1. Preliminaries

For all standard notions, see Engelking [4] or Kuratowski [7]. $A\approx B$ means that A and B are homeomorphic. All metrics are denoted by d and assumed to be bounded by 1; the diameter of a set A is denoted by $\operatorname{diam}(A)$. Q and P denote the space of rationals and irrationals, respectively; 2^ω is the Cantor set.

A subset of a space X is *clopen* if it is both closed and open in X. A space X is *homogeneous* if for each $x, y\in X$, there exists $h:X\approx X$ such that $h(x)=y$; and *strongly homogeneous* if $U\approx X$ for each nonempty clopen subset U of X. A zero-dimensional strongly homogeneous space is homogeneous. If $\mathscr{P}$ and $\mathscr{Q}$ are topological properties, we say that a space X "*is* $\mathscr{P}\cup\mathscr{Q}$" if X can be written as $A\cup B$ with A having property $\mathscr{P}$ and B having property $\mathscr{Q}$. A space X is *nowhere* $\mathscr{P}$ if no nonempty open subset of X is $\mathscr{P}$; and *strongly σ-*$\mathscr{P}$ if $X=\bigcup_{i=0}^{\infty}X_i$, with each X_i closed in X and $\mathscr{P}$. The property $\mathscr{P}$ is said to be *strongly σ-additive* if strongly σ-$\mathscr{P}$ implies $\mathscr{P}$.

By a *complete* space we mean a topologically complete space, i.e., for our spaces, a completely metrizable space, i.e., an absolute G_δ.

I will now briefly review that part of the terminology and results from [2] that is needed for this paper.

1.1. Definition. Let X be a topological space.

(a) *X has property* $\mathscr{S}_k$ if $X=\bigcup_{i=1}^{k}X_i$ with each X_i strongly σ-complete.

(b) For each $k<\omega$,

- *X has property* $\mathscr{P}_{4k}$ if and only if X is $\mathscr{S}_k\cup$ complete;
- *X has property* $\mathscr{P}_{4k+1}$ if and only if X is $\mathscr{S}_{k+1}$;
- *X has property* $\mathscr{P}^1_{4k+2}$ if and only if X is $\mathscr{S}_k\cup$ complete $\cup$ countable;
- *X has property* $\mathscr{P}^1_{4k+3}$ if and only if X is $\mathscr{S}_{k+1}\cup$ countable (also for $k=-1$);
- *X has property* $\mathscr{P}^2_{4k+2}$ if and only if X is $\mathscr{S}_k\cup$ complete $\cup$ σ-compact;
- *X has property* $\mathscr{P}^2_{4k+3}$ if and only if X is $\mathscr{S}_{k+1}\cup$ σ-compact (also for $k=-1$).

1.2. Definition. (a) $\mathscr{X}^1_{-1}=\{Q\}$; $\mathscr{X}^2_{-1}=\{Q\times 2^\omega\}$; $\mathscr{X}_0=\{P\}$.

(b) Let X be zero-dimensional; then for each $k<\omega$,

- $X\in\mathscr{X}_{4(k+1)}$ if and only if X is $\mathscr{P}_{4(k+1)}$, nowhere $\mathscr{P}^2_{4k+2}$, nowhere $\mathscr{P}_{4k+1}$;
- $X\in\mathscr{X}_{4k+1}$ if and only if X is $\mathscr{P}_{4k+1}$, nowhere $\mathscr{P}_{4k}$, nowhere $\mathscr{P}^2_{4(k-1)+3}$;
- $X\in\mathscr{X}^1_{4k+2}$ if and only if X is $\mathscr{P}^1_{4k+2}$, nowhere $\mathscr{P}_{4k}$, nowhere $\mathscr{P}^2_{4(k-1)+3}$;
- $X\in\mathscr{X}^2_{4k+2}$ if and only if X is $\mathscr{P}^2_{4k+2}$, nowhere $\mathscr{P}^1_{4k+2}$, nowhere $\mathscr{P}^2_{4(k-1)+3}$;
- $X\in\mathscr{X}^1_{4k+3}$ if and only if X is $\mathscr{P}^1_{4k+3}$, nowhere $\mathscr{P}^1_{4k+2}$, nowhere $\mathscr{P}_{4k+1}$;
- $X\in\mathscr{X}^2_{4k+3}$ if and only if X is $\mathscr{P}^2_{4k+3}$, nowhere $\mathscr{P}^2_{4k+2}$, nowhere $\mathscr{P}^1_{4k+3}$.

It is easily shown that the above classes are pairwise disjoint.

1.3. Theorem. *Up to homeomorphism, each class $\mathscr{X}_n^{(i)}$ contains a unique space $X_n^{(i)}$; if n is odd, then $X_n^{(i)}$ is first category, if n is even, then it is Baire; each is strongly homogeneous, whence homogeneous.*

Together with 2^ω, $\omega \times 2^\omega$, and the discrete spaces, these spaces are exactly those homogeneous zero-dimensional absolute Borel sets that are finite boolean combinations of complete spaces. The space X_1 is $Q \times P$, the space X_2^1 is van Douwen's T (see [3]), and X_2^2 is van Mill's S (see [11]). Furthermore, $Q \times X_{2n}^{(i)} \approx X_{2n+1}^{(i)}$.

The following lemma is [2, Lemmas 3.4.4(a) and 3.4.8(b)], and will be used in the proof of Lemma 4.6.

1.4. Lemma. (a) $X \approx X_{4k+2}^1$ *if and only if* X *is* $\mathscr{P}_{4k+2}^1$ *and nowhere* $\mathscr{P}_{4k+1}$.

(b) *Let X be compact, and let A be a subset of X. Then for each $k < \omega$, A is $\mathscr{P}_{4k}$ if and only if $X - A$ is $\mathscr{P}_{4(k-1)+3}^2$.*

Only standard, elementary notions from the theory of topological groups will be used; everything can be found in Hewitt and Ross [5] or Comfort [1].

2. Baire Borel sets

Each of the spaces $T = X_2^1$ and $S = X_2^2$ is the union of a complete and a σ-compact subspace, but is nowhere σ-compact; thus, the complete part is dense whence the space is Baire. However, neither space is complete itself. As the results from [2] show, there are many homogeneous Baire zero-dimensional absolute Borel sets, most of them not complete. As the following theorem shows, none of these is a topological group.

2.1. Theorem. *Let X be an analytic Baire topological subgroup of a complete topological group Y. Then X is closed in Y; in particular, X is complete.*

Proof. By a result of Levi [8] (see also [2, 1.12]), X contains a dense complete subspace A. Then $\bar{X} - A$ is first category since A is a dense G_δ in $\bar{X}$. But since $\bar{X}$ is a subgroup of Y, if $x \in \bar{X} - X$, then xA is a dense complete subset of $\bar{X} - X \subset \bar{X} - A$, a clear contradiction. □

Thus, the only analytic Baire zero-dimensional topological spaces that admit the structure of a topological group are the finite spaces, ω, 2^ω, $\omega \times 2^\omega$ and ω^ω. Note that Theorem 2.1 also implies that neither ω, $\omega \times 2^\omega$, nor ω^ω can be embedded as a subgroup of 2^ω. Of the finite spaces, clearly those of cardinality 2^n are subgroups of 2^ω, and it is easy to show that they are the only ones.

3. Borel groups

In this section I will show that if X is a homogeneous first category zero-dimensional absolute Borel set, and $X \approx X^2$, then X can be embedded as a topological subgroup of 2^ω. The concept of independent set (Mycielski [13]) is used. My construction is similar to one in Mauldin [10]; however, I have to be more careful in the computation of the descriptive complexity of the resulting subspace.

3.1. Definitions. *A subset A of 2^ω is independent if for each finite set $\{a_1, \ldots, a_n\}$ of distinct elements of A, $\sum_{i=1}^n a_i \neq 0$.*

The following lemma is well known; however, I do not think the short proof I am going to give has ever appeared in print.

3.2. Lemma. *2^ω contains an independent set $A \approx 2^\omega$.*

Proof. Let $\mathscr{B}$ be the standard basis for 2^ω consisting of sets $B_s = \{x \in 2^\omega : s < x\}$ for $s \in 2^{<\omega}$, and let $f: 2^\omega \to 2^\omega = 2^{2^{<\omega}}$ be the canonical embedding obtained by using this basis (see Engelking [4, Theorem 6.2.16]), i.e., $f(x)_s = 1$ iff $x \in B_s$. Then $A = f[2^\omega]$ works: if $a_0, \ldots, a_n \in A$ are all distinct, say $a_i = f(x_i)$, then there exists s such that $x_0 \in B_s$, but $x_i \notin B_s$ for $i \neq 0$. Thus, $(\sum_{i=0}^n a_i)_s = 1$. □

The following lemma is implicit in the results of [2].

3.3. Lemma. *Let X be a homogeneous first category zero-dimensional absolute Borel set. Then $X \approx Q \times X$.*

3.4. Theorem. *Let X be a homogeneous first category zero-dimensional absolute Borel set such that $X \approx X^2$. Then X can be embedded as a topological subgroup of 2^ω.*

Proof. By Lemma 3.2, 2^ω contains an independent Cantor set A. Embed X densely in A, and let $Y = \langle X \rangle = \bigcup_{n=1}^\infty (\sum_{i=1}^n X_i)$. We claim that $Y \approx X$.

Let $h: 2^\omega \approx A$, and for $s \in 2^{<\omega}$ put $A_s = h[\{x \in 2^\omega : s < x\}]$, $X_s = A_s \cap X$. Note that $X_s \approx X$ since X is in fact strongly homogeneous [2, Corollary 4.4.6]. Let $\phi_n : (2^\omega)^n \to 2^\omega$ be addition ($n \geq 2$), $\phi_1 = \mathrm{id}$. Choose $s_1, \ldots, s_n$ distinct elements of $2^{<\omega}$ of equal length, and consider the corresponding pairwise disjoint $A_{s_1}, \ldots, A_{s_n}$.

Claim 1. *$\phi_n | \prod_{i=1}^n A_{s_i}$ is injective; thus, $\sum_{i=1}^n A_{s_i} \approx \prod_{i=1}^n A_{s_i}$ and $\sum_{i=1}^n X_{s_i} \approx \prod_{i=1}^n X_{s_i} \approx X^n \approx X$.*

Indeed, suppose $\phi_n | \prod_{i=1}^n A_{s_i}$ is not injective, then we may without loss of generality assume that n is minimal with the property that such s_i exist; note that $n > 1$. Thus, there exist $a_i, b_i \in A_{s_i}$ with $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, i.e., $\sum_{i=1}^n a_i + \sum_{i=1}^n b_i = 0$. Since all a_i are distinct and all b_i are distinct, and since A is independent, we must have $a_k = b_l$ for some k, l. In fact, $k=l$ since the A_{s_i} are pairwise disjoint. But then $\sum_{i\neq k} a_i = \sum_{i\neq k} b_i$, so $\phi_{n-1}|\prod_{i\neq k} A_{s_i}$ is not injective, contradicting minimality of n.

Claim 2. $(\sum_{i=1}^n A_{s_i}) \cap Y = \sum_{i=1}^n X_{s_i}$; *thus,* $\sum_{i=1}^n X_{s_i}$ *is closed in* Y.

Indeed, the inclusion "$\supset$" is clear. For "$\subset$", choose $a_i \in A_{s_i}$ with $\sum_{i=1}^n a_i \in Y$, say $\sum_{i=1}^n a_i = \sum_{j=1}^k x_j$, with all x_j distinct (note that $\sum_{i=1}^n a_i \neq 0$). Let $E = \{j\colon x_j \in \{a_1, \ldots, a_n\}\}$, and for each $j \in E$ pick i_j with $x_j = a_{i_j}$; put $F = \{i_j\colon j \in E\}$. Then $\sum_{i\notin F} a_i + \sum_{j\notin E} x_j = 0$. Since A is independent, and all terms in this expression are distinct, if any, we must in fact have $F = \{1, \ldots, n\}$, i.e., each $a_i \in X \cap A_{s_i} = X_{s_i}$.

Claim 3. $\sum_{i=1}^n X_{s_i}$ *is nowhere dense in* Y.

Indeed, let $x_i \in X_{s_i}$ and choose $y \in X - \{x_1, \ldots, x_n\}$. Let $(y_i)_i$ be a sequence of distinct elements of $X - \{x_1, \ldots, x_n, y\}$ converging to y. Then

$$\sum_{i=1}^n x_i + y_j + y_{j+1} \to \sum_{i=1}^n x_i + y + y = \sum_{i=1}^n x_i \quad (j \to \infty).$$

The usual "independence argument" yields that $\sum_{i=1}^n x_i + y_j + y_{j+1} \notin \sum_{i=1}^n X_{s_i}$.

Now note that

$$Y - \{0\} = \bigcup_{k=0}^{\infty} \bigcup_{n=1}^{\infty} \bigcup \left\{ \sum_{i=1}^n X_{s_i}\colon s_1, \ldots, s_n \text{ distinct of length } k \right\};$$

so, from the claims, we see that $Y - \{0\}$ is a countable union of closed nowhere dense copies of the strongly homogeneous space X. By Ostrovskiĭ [14] (see [2, Theorem 3.2.4]), $Y - \{0\} \approx Q \times X \approx X$ by Lemma 3.3. Finally, to show that $Y \approx X$, take an arbitrary $x \in X$, and strictly decreasing local bases $(U_n)_n$, $(V_n)_n$ in Y at 0 and in X at x, respectively, with $U_0 = Y$, $V_0 = X$. For each $n \in \omega$, $U_n - U_{n+1} \approx X \approx V_n - V_{n+1}$, say via h_n, by strong homogeneity. Now define $h\colon Y \approx X$ by $h(0) = x$, $h|(U_n - U_{n+1}) = h_n$. $\square$

I am indebted to van Mill for his comments on an earlier (more complicated) version of the above proof.

Since there are many noncomplete Baire homogeneous zero-dimensional absolute Borel sets X such that $X \approx X^2$ (for example, the countable infinite product of T; as follows from their characterizations, $X \approx X^2$ for *any* zero-dimensional homogeneous absolute Borel set X of exact additive or multiplicative class), it is a consequence of Theorems 2.1 and 3.4 that there exist homogeneous nongroups X such that $Q \times X$ is a topological group, in fact a subgroup of 2^ω!

4. First category Borel nongroups

Recall the definitions of $\mathcal{P}_n^{(i)}$ and $X_n^{(i)}$ from Section 1. In view of the results of the previous section, let me first show:

4.1. Proposition. *For $n \geq 2$, $X_n^{(i)} \not\approx (X_n^{(i)})^2$.*

Proof. Write $X = X_n^{(i)}$. First suppose that n is even. Then since X is not complete, it contains a closed copy of Q (Hurewicz [6]), whence X^2 contains a closed copy of $Q \times X \approx X_{n+1}^{(i)}$; thus, X^2 is not $\mathcal{P}_n$ (Definition 1.2), so $X \not\approx X^2$. If n is odd, let $Y = X_{n-1}^{(i)}$; then $X \approx Q \times Y$. Since n is odd ≥ 2, also $n-1 \geq 2$ so $Y \not\approx Y^2$, say $Y^2 \approx X_k^{(j)}$ (note that Y^2 is a finite boolean combination of complete spaces, since Y is!), $((j), k) \neq ((i), n-1)$. But then $X^2 \approx Q \times Y^2 \approx X_{k+1}^{(j)} \not\approx X_n^{(i)} \approx X$. □

The main result I wiii prove in this section is the following:

4.2. Theorem. *Let X be an absolute Borel set which is the union of a $\mathcal{P}_{4k+1}$ subspace and a countable subspace, but is nowhere $\mathcal{P}_{4k+1}$. Then X does not admit the structure of a topological group.*

4.3. Corollary. *For each $k \in \omega$, X_{4k+3}^1 is a first category homogeneous absolute Borel set which is not a topological group. In particular, $X_3^1 \approx Q \times T$ is not a group.*

The proof of Theorem 4.2 uses a construction originally due to Hurewicz [6], who showed that if X is a noncomplete Borel set in a compact space Z, then Z contains a compactum K with $K \cap X \approx Q$ and $K - X \approx P$. Other "Hurewicz-type theorems" have been proved by Saint-Raymond [15], van Engelen and van Mill [3], van Engelen [2], and recently by Louveau and Saint-Raymond [9]; Steel [16] also contains related results. In Lemma 4.6, I will show that if X (as in Theorem 4.2) is a topological group, then it contains a closed copy Y of X_{4k+2}^1 with a certain special property with respect to the topological group structure of X; in the proof of this "Hurewicz-type theorem" we will actually *use* the results of Louveau and Saint-Raymond mentioned above.

On the other hand, the following lemma shows that such a Y does *not* exist, yielding the required contradiction.

4.4. Lemma. *Let X be a topological group, $X = A \cup S$, with A $\mathcal{P}_{4k+1}$, and S a subgroup. Then X does not contain a closed copy Y of X_{4k+2}^1 such that for some $y \in Y - S$ and all $z \in Y - S$, if $y \neq z$, then $yz \notin S$.*

Proof. Suppose such Y, y exist. Let Z be a proper clopen subset of Y, not containing y; then $Z \approx X_{4k+2}^1$. We claim that $yZ \cap S = \emptyset$. Indeed, if $z \in Z \cap S$, then $z^{-1} \in S$ so if $yz \in S$, then also $yzz^{-1} = y \in S$ since S is a subgroup; and if $z \in Z - S$, then $z \in Y - S$, $y \neq z$, so $yz \notin S$ by assumption. We conclude that yZ is a closed subset of A, whence X_{4k+2}^1 is $\mathcal{P}_{4k+1}$, a contradiction. □

In the proof of Lemma 4.6 we need the following special case of [9, Corollary 8].

4.5. Lemma. *Let X be a Borel subset of a complete space Z, and suppose X is not $\mathcal{P}_{4k+1}$. Then Z contains a Cantor set K such that $K \cap X \approx X^1_{4(k-1)+3}$ and $K - X \approx X_{4k}$.*

Proof. Let B be a dense copy of X_{4k} in 2^ω with $2^\omega - B \approx X^1_{4(k-1)+3}$, and let $\Gamma = D^Z_{2(k+1)}(\Sigma^0_2)$ (i.e., $D \in \Gamma$ iff D is $\mathcal{P}_{4k+1}$ iff $D = \bigcup_{i\,\mathrm{odd}<2(k+1)} (D_i - D_{i-1})$ for some increasing sequence $(D_i)_i$ of F_σ-sets in Z).

Claim. *There exists an increasing sequence $(B_i)_{i<2(k+1)}$ of σ-compact subsets of 2^ω such that $B = \bigcup_{i\,\mathrm{odd}<2(k+1)} (B_i - B_{i-1})$, and B_0 is countable.*

Indeed, since $2^\omega - B$ is $\mathcal{P}^1_{4(k-1)+3}$ is $\mathcal{P}_{4(k-1)+1} \cup$ countable, we can find an increasing sequence $(D_i)_{i<2k}$ of σ-compact subsets of 2^ω, and a countable set C, such that $2^\omega - B = \bigcup_{i\,\mathrm{odd}<2k} (D_i - D_{i-1}) \cup C$. Define $D_{-1} = \emptyset$, $D_{2k} = 2^\omega$, and put $B_{2k+1} = 2^\omega$, $B_i = D_{i-1}$ for i odd$<2k$, and $B_i = D_{i-1} \cup (C \cap D_i)$ for i odd$<2k$. Then $(B_i)_{i<2(k+1)}$ is as required.

Now, in the terminology of [9], if we put $(A_0, A_1) = (Z - X, X)$, we have $X \notin \Gamma$ so $Z - X \notin \check{\Gamma}$, i.e., $\check{\Gamma}$ does not separate (A_0, A_1). Thus, from [9, Corollary 8], (A_0, A_1) H-reduces B, i.e., there exists a one-to-one continuous $f: 2^\omega \to Z$ such that $B = f^{-1}[A_0]$. Put $K = f[2^\omega]$, then $K \cap X = f[2^\omega - B] \approx X^1_{4(k-1)+3}$, and $K - X = f[B] \approx X_{4k}$. $\square$

We now come to the central lemma of this paper. Together with Lemma 4.4, it clearly implies Theorem 4.2.

4.6. Lemma. *Let X be a topological group which is the union of a $\mathcal{P}_{4k+1}$ subspace and a countable subspace, and suppose X is not $\mathcal{P}_{4k+1}$. Then there exists a $\mathcal{P}_{4k+1}$ subspace A of X, a countable subgroup S of X, a closed copy Y of X^1_{4k+2} in X, and a $y \in Y - S$ such that $X = A \cup S$, and for each $z \in Y - S$, if $y \neq z$, then $yz \notin S$.*

Proof. For simplicity let $\mathcal{P} = \mathcal{P}_{4k+1}$; note that $\mathcal{P}$ is closed hereditary and strongly σ-additive.

Write $X = A \cup \tilde{S}$, where A is $\mathcal{P}$ and $\tilde{S}$ is countable; then $S = \langle \tilde{S} \rangle$ is a countable subgroup of X. Embed X as a topological subgroup of a complete topological group Z, and let $\phi: P \to X$ be a continuous surjection. Put

$$W = \{x \in P\text{: there exists a neighborhood } V_x \text{ of } x \text{ in } P\text{, and an } F_\sigma\text{-set } E_x \text{ of } Z\text{, such that } \phi[V_x] \subset E_x\text{, and } E_x \cap X \text{ is } \mathcal{P}\}.$$

Then W is open in P, so there exist countably many open V_i in P, and F_σ-sets E_i in Z, such that $W = \bigcup_{i=0}^\infty V_i$, $\phi[V_i] \subset E_i$, and $E_i \cap X$ is $\mathcal{P}$. Put $E = \bigcup_{i=0}^\infty E_i$, and $E_i = \bigcup_{j=0}^\infty E^i_j$, with E^i_j closed in Z. Then $E^i_j \cap X$ is closed in X, and $\mathcal{P}$, so $E \cap X = \bigcup_{i,j=1}^\infty (E^i_j \cap X)$ is strongly σ-$\mathcal{P}$ whence $\mathcal{P}$. Thus, $X \not\subset E$ whence $F = P - \phi^{-1}[E] \neq \emptyset$.

Claim 1. *If U is a nonempty open subset of F, and G is closed in Z containing $\phi[U]$, then $G\cap X$ is not $\mathcal{P}$.*

Indeed, otherwise, if $U=U'\cap F$ with U' open in $\mathcal{P}$, we would have

$$\phi[U']=\phi[U]\cup\phi[U'-U]\subset E\cup G,$$

and

$$(E\cup G)\cap X=\bigcup_{i,j=0}^{\infty}(E_j^i\cap X)\cup(G\cap X)$$

is strongly σ-$\mathcal{P}$ whence $\mathcal{P}$. This contradicts $U'\not\subset W$.

From this claim, it follows that if U is any nonempty open subset of F, then $\overline{\phi[U]}\cap X$ is not $\mathcal{P}$, and hence by Lemma 4.5, $\overline{\phi[U]}$ contains a Cantor set K such that $K\cap X\approx X^1_{4(k-1)+3}$ and $K-X\approx X_{4k}$.

Put $S=\bigcup_{i=0}^{\infty}S_i$, each S_i finite. For each $s\in\omega^{<\omega}$, we will construct Cantor sets K_s, open subsets U_s of Z, and open subsets W_s of F, such that the following hold.

(1) $K_s\subset\overline{\phi[W_s]}\subset U_s$,
(2) for each $n<\omega$: $\bar{U}_{s\frown n}\cap K_s=\emptyset$,
(3) for each $n, m<\omega$: $\bar{U}_{s\frown n}\cap\bar{U}_{s\frown m}=\emptyset$ if $n\neq m$,
(4) for each $n<\omega$: $\mathrm{Cl}_F(W_{s\frown n})\subset W_s$,
(5) for each $n<\omega$: $\bar{U}_{s\frown n}\subset U_s$,
(6) $\mathrm{diam}(W_s)\leq 2^{-|s|}$ w.r.t. some complete metric on F,
(7) $\mathrm{diam}(U_s)\leq 2^{-\nu(s)}$,
(8) for each $n<\omega$: $d(K_s, K_{s\frown n})\leq 2^{1-\nu(s\frown n)}$,
(9) $K_s\cap X\approx X^1_{4(k-1)+3}$ and $K_s-X\approx X_{4k}$,
(10) K_s is nowhere dense in $K_s\cup\bigcup_{n=0}^{\infty}K_{s\frown n}$,
(11) $B_i=\bigcup\{K_s: |s|\leq i\}\approx 2^{\omega}$,
(12) for each $i\geq 1$, if $s\neq t$, $|s|=|t|=i$, $x\in\phi[W_s]$, $y\in\phi[W_t]$, then $xy\notin\bigcup_{j<i}S_j$,
(13) for each $i\geq 1$, if $|s|=i$, $x\in\phi[W_s]$, $y\in B_{i-1}$, then $xy\notin\bigcup_{j<i}S_j$.

We proceed by induction on $|s|$ and i. First, we put $W_\emptyset=F$, $U_\emptyset=Z$. Then $\overline{\phi[W_\emptyset]}$ contains a Cantor set $K_\emptyset$ such that $K_\emptyset\cap X\approx X^1_{4(k-1)+3}$ and $K_\emptyset-X\approx X_{4k}$. Then (1), (2), (7), (8), and (11) are satisfied.

Next, assume that K_s, U_s, and W_s have been defined for $|s|\leq k$ such that (1)-(13) hold ((11)-(13) for $i\leq k$). Let $s\in\omega^{<\omega}$, $|s|=k$.

Claim 2. *K_s is nowhere dense in $K_s\cup\phi[W_s]$.*

For the easy proof, see [2, p. 40].

Using [2, Lemma 3.3.3], we obtain a countable discrete subset $D_s=\{y_{s\frown n}: n<\omega\}$ of $\phi[W_s]-K_s$ such that $\bar{D}_s=D_s\cup K_s$, and $d(y_{s\frown n}, K_s)\leq 2^{-\nu(s\frown n)}$ for each $n<\omega$. Let $U_{s\frown n}$ be a neighborhood of $y_{s\frown n}$ in X such that $\bar{U}_{s\frown n}\cap K_s=\emptyset$, $\bar{U}_{s\frown n}\cap\bar{U}_{s\frown m}=\emptyset$ if $n\neq m$, $\mathrm{diam}(U_{s\frown n})\leq 2^{-\nu(s\frown n)}$, and $\bar{U}_{s\frown n}\subset U$ for each $n, m<\omega$. Since $y_{s\frown n}\in\phi[W_s]$, $y_{s\frown n}=\phi(x_{s\frown n})$ for some $x_{s\frown n}\in W_s$; hence there is an open neighborhood $W'_{s\frown n}$ of $x_{s\frown n}$ in F such that $\mathrm{Cl}_F(W'_{s\frown n})\subset W_s$, $\mathrm{diam}(W'_{s\frown n})\leq 2^{-|s|-1}$, and $\overline{\phi[W'_{s\frown n}]}\subset U_{s\frown n}$. Now exactly the same proof as in [2, Lemma 3.3.4] shows that if $W_{s\frown n}$ is *any* nonempty open subset of $W'_{s\frown n}$, and $K_{s\frown n}\subset\overline{\phi[W_{s\frown n}]}$ is a Cantor set such that $K_{s\frown n}\cap X\approx X^1_{4(k-1)+3}$

and $K_{s^\frown n} - X \approx X_{4k}$, then hypotheses (1)-(10) are satisfied. Hypothesis (11) also holds: indeed, if $\mathcal{U}$ is an open covering of B_{k+1}, then there exists a finite $\mathcal{V} \subset \mathcal{U}$ which covers B_k, and hence there exists $\varepsilon > 0$ such that $B^k_\varepsilon = \{z \in Z: d(z, B_k) \leq \varepsilon\} \subset \bigcup \mathcal{V}$. Let $M^\varepsilon_k = \{s \in \omega^{<\omega}: |s| = k+1, K_s \subset B^\varepsilon_k\}$. By (1), (7) and (8), M^ε_k is finite, so $\bigcup\{K_s: s \in M^\varepsilon_k\}$ is compact and hence covered by a finite $\mathcal{W} \subset \mathcal{U}$; then $\mathcal{V} \cup \mathcal{W}$ is a finite subcovering of $\mathcal{U}$.

We now show how to actually define the $W_{s^\frown n}$ such that (12) and (13) are satisfied. To this end, put $\{y_{s^\frown n}: |s| = k, n < \omega\} = \{p_j: j < \omega\}$, faithfully indexed, $p_j = y_{s_j{}^\frown n_j}$. The proof just given to show that (11) holds also yields that $B = B_k \cup \{p_j: j < \omega\}$ is compact. Let $L = \bigcup_{j \leq k} S_j$, and put $C = LB^{-1} \cup B^{-1}L$.

Claim 3. *For each nonempty open subset U of F, $\phi[U] \not\subset C$.*

Indeed, by Claim 1, it suffices to show that $C \cap X$ is $\mathcal{P}$; note that $C \cap X = L(B \cap X)^{-1} \cup (B \cap X)^{-1}L$ since $L \subset X$, i.e.,

$$C \cap X = \bigcup_{j=0}^{\infty} (Lp_j^{-1} \cup p_j^{-1}L) \cup L(B_k \cap X)^{-1} \cup (B_k \cap X)^{-1}L.$$

Now $B_k \cap X = \bigcup\{K_s \cap X: |s| \leq k\}$, where each $K_s \cap X$ is $\mathcal{P}^1_{4(k-1)+3}$ and closed in X; thus, $(B_k \cap X)^{-1}$ is a countable union of $\mathcal{P}^1_{4(k-1)+3}$ subspaces closed in X, hence so is $L(B_k \cap X)^{-1} \cup (B_k \cap X)^{-1}L$ (recall that L is finite). Thus, $C \cap X$ is strongly σ-$\mathcal{P}^1_{4(k-1)+3} \cup$ countable is $\mathcal{P}^1_{4(k-1)+3} \cup$ countable is $\mathcal{P}^1_{4(k-1)+3}$.

We will construct the $W_{s_m{}^\frown n_m}$ inductively, together with open neigborhoods $U^m_{s_j{}^\frown n_j}$ of p_j in Z, for $j > m$, such that of course $W_{s_m{}^\frown n_m} \subset W'_{s_m{}^\frown n_m}$, and furthermore

(a) $\phi[W_{s_m{}^\frown n_m}] \subset \bigcap_{j<m} U^j_{s_m{}^\frown n_m}$,

(b) if $j > m$, $x \in \phi[W_{s_m{}^\frown n_m}]$, $y \in U^m_{s_j{}^\frown n_j}$, then $xy \notin L$, $yx \notin L$,

(c) if $x \in \phi[W_{s_m{}^\frown n_m}]$, $y \in B_k$, then $xy \notin L$.

Clearly, (a) and (b) together imply (12) and (c) implies (13) for $i = k+1$.

Suppose we are done for $m' < m$. Since $p_m = y_{s_m{}^\frown n_m} = \phi(x_{s_m{}^\frown n_m}) \in \bigcap_{j<m} U^j_{s_m{}^\frown n_m}$, $x_{s_m{}^\frown n_m} \in W'_{s_m{}^\frown n_m}$, we can find a nonempty open subset U of F such that $U \subset W'_{s_m{}^\frown n_m}$ and $\phi[U] \subset \bigcap_{j<m} U^j_{s_m{}^\frown n_m}$. Let $z \in \phi[U] - C$, by Claim 3. Since C is compact, we can find disjoint open neighborhoods V, W in Z of z, C, respectively. Let $W_{s_m{}^\frown n_m}$ be open in F and contained in U such that $\phi[W_{s_m{}^\frown n_m}] \subset V$. Then (a) is satisfied. Since $C \subset W$, we have $Lp_j^{-1} \cup p_j^{-1}L \subset W$ for each $j > m$, and hence there is an open neighborhood $U^m_{s_j{}^\frown n_j}$ of p_j in Z such that $L(U^m_{s_j{}^\frown n_j})^{-1} \cup (U^m_{s_j{}^\frown n_j})^{-1}L \subset W$. Then if $x \in \phi[W_{s_m{}^\frown n_m}]$, $y \in U^m_{s_j{}^\frown n_j}$ $(j > m)$, then $Ly^{-1} \cup y^{-1}L \subset W$, so $x \notin Ly^{-1} \cup y^{-1}L$, i.e., $xy \notin L$ and $yx \notin L$, proving (b). For (c), note that $LB_k^{-1} \subset C \subset W$ so $x \notin LB_k^{-1}$ for $x \in \phi[W_{s_m{}^\frown n_m}]$. This completes the construction of the K_s, U_s, and W_s.

Now put $B = \bigcup_{i=0}^{\infty} B_i$, and let $Y = B \cap X$; we claim that Y is as required. We first show that if $x \in B - \bigcup_{i=0}^{\infty} B_i$, then $x \in X$. Indeed, fix $i < \omega$. Since $x \notin B_i$, also $x \notin B^\varepsilon_i$ for some $\varepsilon > 0$. From (1) and (4) it follows that $\bigcup_{j=0}^{\infty} B_j \subset B_i \cup \bigcup_{|s|=i} \overline{\phi[W_s]}$, and from (1), (7) and (8) that $\overline{\phi[W_s]} \subset B^\varepsilon_i$ for all but finitely many $s \in \omega^{<\omega}$ with $|s| = i$. Hence for some finite $M \subset \{s \in \omega^{<\omega}: |s| = i\}$, we have $B \subset B^\varepsilon_i \cup \bigcup_{s \in M} \overline{\phi[W_s]}$. Then $x \in \overline{\phi[W_s]}$ for some $s \in M$, and this s is unique with $|s| = i$ by (1), (3) and (5) (or trivially, if $i=0$). So by (4), there exists $\sigma\in\omega^\omega$ such that $x\in\bigcap_{s<\sigma}\overline{\phi[W_s]}$, which is a one-point set by (1) and (7). Also, $\bigcap_{s<\sigma}\bar{W}_s=\bigcap_{s<\sigma}W_s$ is a one-point set by (6), and by completeness of the metric on F. Hence, $\{x\}=\phi[\bigcap_{s<\sigma}W_s]$, and we conclude that $Y=B-\bigcup_{i=0}^\infty(B_i-X)$.

Since B is complete, and each B_i is closed and nowhere dense in B by (10) and (11), and S is countable, we can choose $y\in B-\bigcup_{i=0}^\infty B_i-S$, say $\{y\}=\phi[\bigcap_{s<\sigma}W_s]$. If z is any element of $Y-S$, $z\neq y$, then either $z\in B_i$ for some i, whence $z\in B_m$ for all $m\geq i$ and $yz\notin\bigcup_{m\geq i}\bigcup_{j<m}S_j=S$ by (13); or $\{z\}=\phi[\bigcap_{s<\tau}W_s]$ for some $\tau\in\omega^\omega$, $\tau\neq\sigma$, whence $\sigma|m\neq\tau|m$ for all $m\geq$ some i and $xy\notin\bigcup_{m\geq i}\bigcup_{j<m}S_j=S$ by (12). It remains to show that $Y\approx X^1_{4k+2}$. We use the characterization of Lemma 1.4(a).

First, note that

$$Y=\bigcup_{i=0}^\infty(B_i\cap X)\cup\left(B-\bigcup_{i=0}^\infty B_i\right)=\bigcup_{s\in\omega^{<\omega}}(K_s\cap X)\cup\left(B-\bigcup_{i=0}^\infty B_i\right)$$

is strongly σ-$\mathcal{P}^1_{4(k-1)+3}\cup$complete is $\mathcal{P}^1_{4(k-1)+3}\cup$complete is $\mathcal{P}^1_{4k+2}$. Finally, take a nonempty clopen subset V of Y and suppose that $V=\bigcup_{i=0}^\infty V_i$, with V_i closed in V (hence in Y) and $\mathcal{P}_{4k}$; since $B-\bigcup_{i=0}^\infty B_i$ is a dense complete subset of Y, V is Baire, so some Int $V_i\neq\emptyset$, so some nonempty clopen subset W of Y is $\mathcal{P}_{4k}$. But $W\cap B_j\neq\emptyset$ for some j, so $W\cap K_s$ is a nonempty clopen subset of $K_s\cap X\approx X^1_{4(k-1)+3}$ for some $|s|\leq j$ which is $\mathcal{P}_{4k}$; then if W' is nonempty and clopen in K_s, $W'\cap X\subset W\cap K_s$, then by Lemma 1.4(b), $W'-X$ is $\mathcal{P}^2_{4(k-1)+3}$, and open nonempty in $K_s-X\approx X_{4k}$, a contradiction; so Y is nowhere $\mathcal{P}_{4k+1}$. □

I do not know whether the above lemma remains true if "countable" is replaced by "σ-compact"; if so, this would imply Theorem 4.2, modified similarly, and thus Corollary 4.3 would hold for the spaces X^2_{4k+3}. Trying to copy the proof for this case, we see that difficulties arise only in the proof of Claim 3: instead of a finite L, we have to deal with a compact L, so the question is whether the (group) product $(K_s\cap X)L$ (which is closed in X) of the $\mathcal{P}^1_{4(k-1)+3}$ space $K_s\cap X$ and the compact L is $\mathcal{P}_{4k+1}$. I do not know the answer, except of course in the trivial case $k=0$, since (σ-compact) compact is σ-compact is strongly σ-complete, yielding:

4.7. Corollary. *$Q\times S$ is not a topological group.*

I do not know if the above proof could be adapted to also deal with the spaces X_{4k+1}. I think that other techniques will be needed to treat the homogeneous Borel sets of higher complexity.

Note added in proof

(1) I was informed by T. Dobrowolski that Theorem 2.1 is implied by an old result of Banach.

(2) In a forthcoming paper I will show, among other things, that the spaces X_{4k+1} and X^2_{4k+3} are not topological groups.

References

[1] W.W. Comfort, Topological groups, in: K. Kunen and J. Vaughan, eds., Handbook of General Topology (North-Holland, Amsterdam, 1984).
[2] F. van Engelen, Homogeneous zero-dimensional absolute Borel sets, Thesis, CWI Tract 27, Amsterdam, 1986.
[3] F. van Engelen and J. van Mill, Borel sets in compact spaces: some Hurewicz-type theorems, Fund. Math. 124 (1984) 271-286.
[4] R. Engelking, General Topology (PWN, Warszawa, 1977).
[5] E. Hewitt and K.A. Ross, Abstract Harmonic Analysis I, Grundlehren der Mathematischen Wissenschaften 115 (Springer, Berlin, 1963).
[6] W. Hurewicz, Relativ perfekte Teile von Punktmengen und Mengen (A), Fund. Math. 12 (1928) 78-109.
[7] K. Kuratowski, Topology I (Academic Press, New York, 1966).
[8] S. Levi, On Baire cosmic spaces, in: General Topology and Its Relations to Modern Analysis and Algebra V, Prague, 1981 (Heldermann, Berlin, 1983) 450-451.
[9] A. Louveau and J. Saint-Raymond, Borel classes and closed games: Wadge-type and Hurewicz-type results, Trans. Amer. Math. Soc. 304 (1987) 431-467.
[10] R.D. Mauldin, On the Borel subspaces of algebraic structures, Indiana Univ. Math. J. 29 (1980) 261-265.
[11] J. van Mill, Characterization of a certain subset of the Cantor set, Fund. Math. 118 (1983) 81-91.
[12] J. van Mill, In memoriam: Eric Karel van Douwen, Topology Appl. 31 (1989) 1-18.
[13] J. Mycielski, Independent sets in topological algebras, Fund. Math. 55 (1964) 139-147.
[14] A.V. Ostrovskiĭ, Continuous images of the Cantor product $C \times Q$ of a perfect set C and the rational numbers Q, Seminar on General Topology, Moskov Gos. Univ., Moscow, 1981, 78-85 (in Russian).
[15] J. Saint-Raymond, La structure Borélienne d'Effros est-elle standard?, Fund. Math. 100 (1978) 201-210.
[16] J.R. Steel, Analytic sets and Borel isomorphisms, Fund. Math. 108 (1980) 83-88.